Supplement I: Gauge Symmetries and Gauge Forces

According to Penrose (2004), all physical interactions are governed by "gauge connections" which depend crucially on spaces having exact symmetries (p. 289). From the perspective of quantum physics, the idea of gauge symmetry has been applied by pioneers such as Schrödinger, Klein, Fock and others (for an overview, see Jackson and Okun, 2001). It is suitable to introduce the realistic force conception by means of a simplified mechanical picture (following Harlander, 2013).

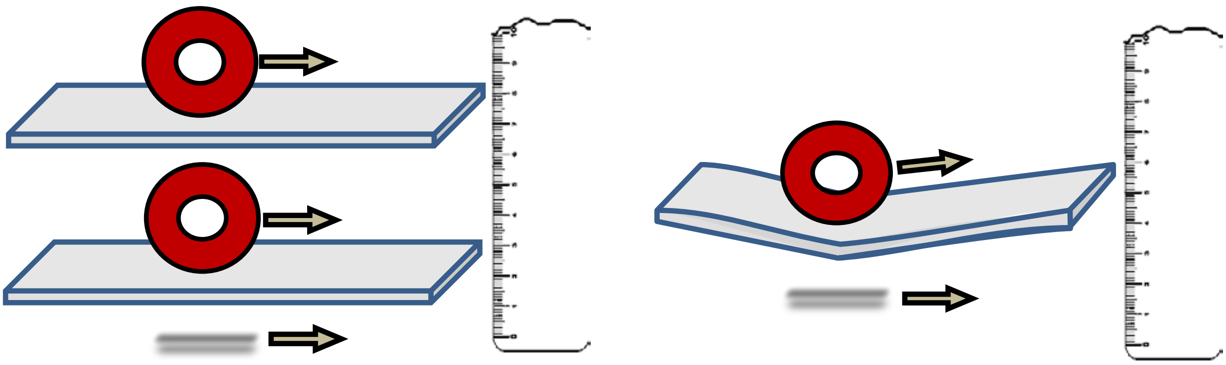

Figure 1: Left: Global Gauge. Right: Local gauge

Figure 1 gives a mechanical example of a so-called gauge symmetry provided by a tire rolling on a pane of glass. The shining sun is producing a moving shadow, which is the essential thing we can observe (similar to Plato's allegory of the cave). For the movement of the shadow the absolute altitude of the pane is not relevant, only the velocity of the rolling tire is. The fact that the whole scenery of the rolling tire can be moved vertically without changing the movement of the shadow corresponds to a *global symmetry*.

Now assume that there is a deformation of the pane resulting in a local change of the altitude of the tire. The variation is producing a global symmetry breaking. The dynamic effect of the symmetry breaking is that the velocity of the tire is changing by means of the deformation. The shadow at the bottom reflects this behaviour.

The request for *local symmetry* is now easy to understand. It refers to the demand that the movement of the shadow does not give any indication for the deformation of the pane. Obviously, this can happen if we slow down or accelerate the tire dependent on the local deformation. In other words, the request of local symmetry demands that we introduce a varying force.[1]

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[1] Of course, pictures such as Figure 1 should be used with great caution. Moses forbid the Israelites to make any image of God. Similarly, in several respects, Dirac remarked that we should not try to make visualizations of quantum theory.

Generally, the idea of founding forces by symmetries is as follows. Assume a physical system is invariant with respect to some global group of continuous transformations (for instance, independence of space/time). Then the idea of gauge invariance is to make the stronger assumption that the basic physical equations describing the system have to be invariant when the group operations are considered locally (i.e., dependent on time and the other coordinates of the system). Normally, this principle of gauge invariance leads to a modification of the original equation and introduces additional terms that can be interpreted as new "forces" induced by the "gauge field", which describe these local dependencies.[2]

The Gauge Manifesto

(i) All musical forces are gauge forces.

(ii) Any gauge force is founded in a symmetry group and a gauge field.

(iii) Tones are modelled by vectors of a 2-dimensional spinor Hilbert space. Hence, the basic symmetry group is the group of unitary transformations.

(iv) The stationary Schrödinger-Pauli equation for spinors $\psi(x)$ is the fundamental equation under discussion. It has the following general form:

$$-\frac{\partial^2\psi(x)}{\partial x^2} + \boldsymbol{M}(x)\cdot\frac{\partial\psi(x)}{\partial x} + \boldsymbol{V}(x)\cdot\psi(x) = 0. \qquad (1)$$

Hereby, the matrix function $\boldsymbol{M}(x)$ describes the magnetic vector potential, and the matrix function $\boldsymbol{V}(x)$ describes the scalar potentials of the electrostatic force (Note that this force is functionally equivalent to the force of gravity).

(v) A special case of the Schrödinger-Pauli equation is the *free* equation:

$$-\frac{\partial^2\psi(x)}{\partial x^2} = E\,\psi(x). \qquad (2)$$

The free equation corresponds to the general equation (2) with $\boldsymbol{M}(x) = 0$ and $\boldsymbol{V}(x) = -E$. A simple solution is $\psi(x) = \frac{1}{\sqrt{\pi}}\begin{pmatrix}\cos x/2\\ \sin x/2\end{pmatrix}$, for $E = 1/4$.

(vi) All gauge forces result from gauge transformations.

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[2] The idea of gauge invariance was first developed by Hermann Weyl in 1918, when he made the attempt to unify gravity and electromagnetism. Weyl assumed that the length of any single vector is arbitrary. Only the relative lengths of any two vectors and the angle between them are preserved under parallel transport. This was the birth of a new idea in physics which was called "gauge invariance" by Weyl. Even when Weyl's attempt to develop a unified theory failed, the idea survived and was extremely successful later on. It is this success that justifies the theory. The theory itself remains mysterious to a certain degree: we do not have an independent, physical or methodological motivation for it.

(vii) A gauge transformation is a representation of the fundamental symmetry group (or one of its subgroup). It is specified by a particular gauge field.

(viii) The forces specified in the Hamiltonian by the matrix fields $\boldsymbol{M}(x)$ and $\boldsymbol{V}(x)$ are *founded* by a gauge transformation if it transforms all solutions of the stationary Schrödinger-Pauli equation (1) (with specified matrix functions $\boldsymbol{M}(x)$ and $\boldsymbol{V}(x)$) into a force-free solution of (2).

(ix) *Deformation forces* and *phase forces* are defined by particular gauge transformations. Deformation forces result from the rotation group SO(2) and the gauge field $\vartheta(x)$ of rotation. Phase forces result from the unitary group U(1) and the gauge field of phase shifts $\varphi(x)$. The combination of both gauges is possible.

(x) Other symmetry groups relevant for tonal music are transposition symmetry (defined by the cyclic group) and particular modulation groups investigated by Mazzola.

Electromagnetic theory as gauge theory

Electromagnetic theory is historically the first and also the simplest example of gauge theory. In the following, we demonstrate the connection between gauge transformation and resulting forces based on a phase gauge. In this case, the Schrödinger equation is our starting point (with scalar wave function $\psi(x)$).

In quantum physics, gauge symmetry is essentially related to the distinction between overt and covert physical quantities. The value of a wave function $\psi(x)$ is covert as it cannot be directly observed in physical measurement. Only probabilities $p(x) = |\psi(x)|^2$ and expectation values are measurable. Hence, the probabilities must be invariant under a shift of the wave function's phases.

Let ψ be a wave function solving the free-particle Schrödinger equation

$$H\,\psi(x) = E\psi(x) \tag{3}$$

With Hamiltonian $H = T = p^2$ where T is the operator of kinetic energy, which is given through the quantum mechanical momentum operator[3]

$$p = -i\frac{\partial}{\partial \mathrm{x}} \tag{4}$$

Furthermore, let $\varphi \in \mathbb{R}$ be a real phase value. Then, the operation $\tilde{\psi} = e^{i\varphi}\psi$ yields another solution of the Schrödinger equation obtained by multiplying Eq. (3) with $e^{i\varphi} \in \mathrm{U}(1)$. Yet, this *global gauge transformation* does not affect the observable probabilities $\tilde{p} = |\tilde{\psi}|^2 = p$.

However, when the phase shift becomes a function of space, we encounter some complication. Then $\varphi(x)$ describes a *local gauge transformation*. Writing

[3] Note that we normalize particular physical quantities such as mass *m*, charge *q*, light speed *c* and Planck's quantum of action $\hbar$ to natural units = 1 here.

$$\tilde{\psi}(x) = e^{i\varphi(x)}\psi(x) \tag{5}$$

with $e^{i\varphi(x)} \in \mathrm{U}(1)$ as a general unitary U(1) gauge transformation, we compute the spatial derivatives in (3):

$$\frac{\partial\tilde{\psi}}{\partial x} = i\frac{\partial\varphi}{\partial \mathrm{x}}e^{i\varphi}\psi + e^{i\varphi}\frac{\partial\psi}{\partial \mathrm{x}} = e^{i\varphi}\left(\frac{\partial}{\partial \mathrm{x}} + i\frac{\partial\varphi}{\partial \mathrm{x}}\right)\psi \tag{6}$$

Repetition of the derivation yields the Laplacean

$$\frac{\partial^2\tilde{\psi}}{\partial x^2} = \frac{\partial}{\partial x}\left[e^{i\varphi}\left(\frac{\partial}{\partial \mathrm{x}} + i\frac{\partial\varphi}{\partial \mathrm{x}}\right)\psi\right] = e^{i\varphi}\left(\frac{\partial}{\partial \mathrm{x}} + i\frac{d\varphi}{dx}\right)^2\psi \tag{7}$$

For the operator appearing in round brackets we introduce the notation

$$\mathrm{D}_x = \frac{\partial}{\partial \mathrm{x}} + i\frac{\partial\varphi}{\partial \mathrm{x}} \tag{8}$$

that is called the *covariant derivative* for continuous U(1) gauge symmetry. The opposite gradient of the phase function $\varphi(x)$ is called the gauge field $A(x)$:

$$A(x) = -\frac{\partial\varphi}{\partial \mathrm{x}}\ . \tag{9}$$

In Maxwellian electrodynamics the (external) magnetic flux $\mathcal{B}$ together with the (external) electrostatic field $\mathcal{E}$ contribute to the Lorentz force that is exerted to a charged particle *q* moving with velocity *v*

$$F = q(\mathcal{E} + v \times \mathcal{B}). \tag{10}$$

In U(1) gauge theory, the Lorentz force is straightforwardly obtained by inserting the gauge field $A(x)$ into the covariant derivative (8), thereby substituting the momentum operator through the *minimal coupling*

$$\tilde{p} = -i\mathrm{D}_x$$

$$\tilde{p} = -i\frac{\partial}{\partial \mathrm{x}} + \frac{\partial\varphi}{\partial \mathrm{x}}$$

$$\tilde{p} = p - A(x)\ . \tag{11}$$

Inserting this into the Schrödinger equation (3) yields

$$(p - A)^2\psi(x) = E\psi(x) \tag{12}$$

Expanding the bracket then entails

$$(p - A)(p - A)\psi(x) = E\psi(x)$$

$$(p^2 - pA - Ap + A^2)\psi(x) = E\psi(x)$$

$$p^2\psi(x) - Ap\psi(x) + A^2\psi(x) = E\psi(x)$$

$$-\psi''(x) + iA\psi'(x) + A^2\psi(x) = E\psi(x) \tag{13}$$

where we have used the electrodynamic Coulomb constraint (Jackson and Okun, 2001) $pA = -iA' = 0$ in stepping from the second to the third line.

Equation (13) is, up to the imaginary factor of $A\psi'(x)$, identical with the Schrödinger equation (1), thereby justifying the realistic interpretation of the first derivative of the wave function in the deformation model (cf. Eqs. 14, 15 in the Manuscript) with musical magnetic force. The operator $\boldsymbol{M}$ hence reflects the role of the magnetic vector potential A. Moreover, the term $\boldsymbol{V}(x) = A^2$ acts as a scalar operator on the wave function. Therefore its realistic interpretation must be that of the gauged electrostatic potential.

A plausible choice for $A(x)$ is a function proportional to $\sqrt{\frac{1}{x}}$. According to Eq. (9), this requires a gauge function $\varphi(x)$ that is proportional to $\sqrt{x}$. With this choice we get an electrostatic potential proportional to $\frac{1}{x}$ and an electrostatic force potential to $\frac{1}{x^2}$. This corresponds to the well-known Coulomb force, proportional to the inverse square of the distance between two point charges. Hence, a particular choice of the gauge function gives particular specifications of both the electrostatic and the magnetic forces.

Note, however, that physical magnetism involves the imaginary unit in Eq. (13) which is absent in musical magnetism. Thus, it is not possible to interpret this metaphorically in the sense of Larson (2012).

We stress again, that for physicists, who are familiar with gauge theory, the terms "magnetism" and "electrostatics" conform to structural properties of the gauged Schrödinger equation. As clarified by Eq. (1), the magnetic potential $\boldsymbol{M}$ refers to the factor of the first derivation of the wave function, i.e. $\frac{\partial \psi(x)}{\partial x}$, whereas the electrostatic potential $\boldsymbol{V}$ refers to the factor of the wave function $\psi(x)$ itself. Depending on the underlying symmetry group and the particular gauge function, these structural properties turn into predictions that can be tested empirically.

Consonance/dissonance: A case for asymmetric deformation

A very prominent phenomenon is the occurrence of graded consonance/dissonance. In an indirect way, it relates to static attraction. According to Parncutt (1989), the degree of (musical) consonance of a chord is linked to the distribution of potential root tones of a chord. Hereby, the root tone can be seen as the tone in the case of the maximum static attraction given the chord as musical context. In cases with a single, prominent root tone, the chord sounds more consonant than when several root tones are in competition. Formally, we can explain the degree of consonance of a chord as the static attraction value of the (root) tone with maximum attraction after normalizing the attraction profile (i.e., the attraction values of the 12 tones sum up to 1).

The mirror symmetry (around the triton) of the spatial deformation model leads to important problems when accounting for the differences between major and minor modes. Important distinctions between major and minor modes were discussed already 90 years ago (Heinlein, 1928). Recently, Johnson-Laird, Kang, & Leong (2012) have investigated chords including major triads (CEG), minor triads (CE♭G), diminished triads (CE♭G♭), and augmented triads (CEG#). The following table shows the empirical ratings of the chord's consonance. Clearly, the major chords exhibit the highest degree of consonance followed by the minor chords. Further, the diminished chords are ranked lower and, at the bottom, we (surprisingly) find the augmented chords. It is not difficult to see that the hierarchical model and the symmetric deformation model predict the same degrees of consonance for major and minor chords.

Table 1: Empirical rankings and model predictions for common triads. The predictions of the models concern the strength of the tone with strongest static attraction using normalized attraction profiles. The symmetry breaking is provided by a weakly interfering phase gauge field (2 %). It gives the asymmetry between major and minor.

Triad Class	*Empirical Consonance Rating*	*Hierarchical Model*	*Deformation Model*	*SU(2) Combined Model*
major	5.33	.49	.49	.495
minor	4.59	.49	.49	.49
diminished	3.11	.34	.36	.34
augmented	1.74	.33	.34	.33

In order to model the ranking of the three triadic chords considered in Table 1, we have considered a modification of the deformation model. Here, we combine the deformation model with a weakly coupled phase field. It is expected that even a weak coupling leads to interfering terms that can provide the desired asymmetry. It should be stressed that this idea does not introduce any new parameter and is valid for a whole variety of weak couplings (from 0.1% – 8%).

The following pictures show the increase of asymmetry with rising coupling (left 3%, middle 6 %, right 8 %). We stress that it is not the intent of this presentation to give a verification of our theory. Rather, it is a preliminary illustration of how certain asymmetries of static attraction, including the asymmetry between major and minor, could find a natural place within the present theory.

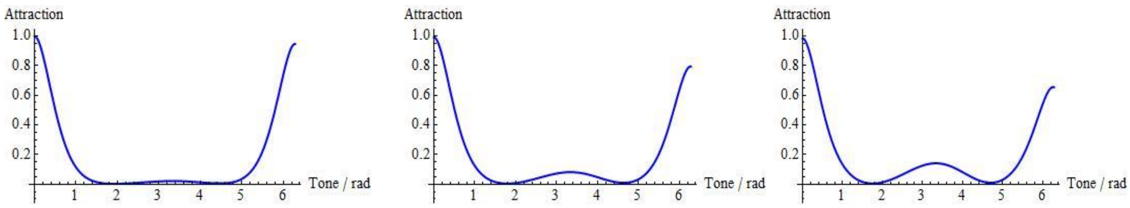

Figure 2: Kernel functions for the combined model with a very weak coupling of the phase gauge field: 3 % on the left hand side, 6 % in the middle, and 8 % and the right hand side.

It is evident that the weak coupling of the phase field does not only break the mirror symmetry relative to the tritone but also octave equivalence (the higher tonic gets a lower degree of attraction than the lower). This fact may make sense considering the different consonance values for different inversions of a given chord. We cannot further pursue this issue here.

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Supplement II: The symmetry Group SU(2)

The group SU(2) is the set of all two dimensional, complex unitary matrices $\boldsymbol{U}$ with unit determinant.

As outlined in the Manuscript, Sect. 4.1, this group can be generated by the Pauli matrices (1):

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (1)$$

In the general form, the unitary matrix $\boldsymbol{U}$ is given by the following generating expression:

$$\boldsymbol{U} = e^{-\frac{i}{2}\sum_{j=1}^{3} \delta_j(x)\cdot\sigma_j} \qquad (2)$$

Hereby, the functions $\delta_1(x)$, $\delta_2(x)$, and $\delta_3(x)$ are real phase functions defining an arbitrary local gauge transformation. Evaluating the matrix exponential in (2) gives the general SU(2) transformation matrix

$$\boldsymbol{U}(\vartheta, \varphi, \tau) = \begin{pmatrix} \cos\vartheta e^{-i\varphi} & -\sin\vartheta e^{i\tau} \\ \sin\vartheta\, e^{-i\tau} & \cos\vartheta e^{i\varphi} \end{pmatrix} \qquad (3)$$

where we distinguish $\vartheta = \delta_1(x), \tau = \delta_2(x)$, $\varphi = \delta_3(x)$.

Modern particle physics assumes four fundamental forces: electromagnetic, the weak, the strong, and the gravitational force. For a comprehensive introduction the reader is referred to Weinberg (1992).

As an example, we consider the electromagnetic force that is required for the description of electrons and positrons. The corresponding theory is developed in *quantum electrodynamics* and can be seen as gauge theory based on the symmetry group U(1) (the set of all one dimensional, complex unitary matrices). The starting point for the development of an appropriate field theory is the Dirac equation. In this case, the multiplication of the wave function with a local phase factor $e^{i\varphi(x,t)}$ introduces an additional term in the transformed Dirac equations which destroys the symmetry described by the unitary group U(1). The crucial idea is to compensate the destroying term by an additional term modifying the original electromagnetic potential. This term is seen as describing an interaction of the original electromagnetic field with a gauge field. Obviously, this idea realizes a new dynamical principle coupling the gauge field with the electromagnetic field of the electron. There is a natural interpretation of the gauge field: it describes the interaction of a *photon* with the electron. In other words, the exchange of a photon is realizing a new force found by the idea of a gauge transformation.

Next, we will consider the weak force, which is responsible for beta decay, for instance. Beta decay produces a neutrino, which does not interact via the strong or electromagnetic force.[1] In the late 1960s, Weinberg, Salam, and Glashow developed a theory explaining the electroweak interaction (cf. Weinberg, 1992). They stipulated that a triplet of massless spin-1 particles is acting as carriers of the force. Two of these particles are charged and one is neutral. These particles are the weak bosons, W^+, W^-, and Z^0.[2]

The three gauge bosons are not directly associated with the Pauli matrices σ_i and the gauge functions $\delta_i(x)$. Rather, we have to construct the isospin ladder operators

$$\sigma_+ = \sigma_1 + i\,\sigma_2 \text{ and } \sigma_- = \sigma_1 - i\,\sigma_2 \tag{4}$$

These linear combination are directly associated with $W^+$and W^-, respectively (called the *charged currents* in particle physics). Only the third Pauli matrix and the gauge function $\varphi = \delta_3(x)$ are directly associated with the gauge boson Z^0 (called the *neutral current*).

For the treatment of the strong force (the short-range attractive force that holds together the nucleus of the atom) the symmetry group SU(3) has been proposed (defined by all three dimensional, complex unitary matrices $\boldsymbol{U}$ with unit determinant). It has eight generators. The corresponding field theory stipulates quarks that come in several different varieties (called flavours and colours). Strong forces do not notice the different flavours of quarks. However, they are depending on colours. Just as the electromagnetic forces are mediated by particular gauge particles (the photons), so we expect that the quark-quark interactions are described in terms of the exchange of a particular gauge particle. These particles are called *gluons* and the eight gluons are explicitly defined within the SU(3) gauge theory.

We have pointed out that the symmetry group SU(2) and the corresponding gauge symmetry is not only applicable in particle physics but also in mathematical musicology. The same cannot be said for the symmetry group SU(3). At the moment, it is rather unclear whether this symmetry group can have any relevance in tonal music (a

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[1] Beta decay are typical examples with W-exchange (charged current): $n \overset{W}{\rightarrow} p + e^- + \bar{\nu}$. Another example is the reaction $\bar{\nu} + p \overset{W}{\rightarrow} e^+ + n$. An example with Z-exchange (neutral current) is here: ${}^2_1D + \nu \overset{Z}{\rightarrow} p + n + \nu$ (neutrino reactions with deuterium decay).

[2] At first, the quantum field theory of electroweak interactions did not seem to have much going for it. There was no model for how the weak bosons acquired their mass. The Z^0 also had not been observed at the time. In the standard model of particle physics, the weak bosons acquire large masses, approximately 86 and 97 times the mass of a proton for the Ws and Z^0, respectively. The fundamental mechanism is a kind of symmetry breaking caused by interaction with the Higgs field (Hey and Walters, 2003).

rather speculative point is that the group SU(3) could be used to formalize the LPR operators in Neo-Riemannian tonal grids).[3]

We think that physics has generated many great ideas that are applicable in mathematical music theory as well. This concerns gauge transformation based on the symmetry groups U(1) and SU(2). Another great idea from physics is the notion of (spontaneous) *symmetry breaking*. It likewise makes sense in the domain of tonal music. However, we should be careful when looking for analogies. In most cases, there is no connection between physical ideas and musical principles. For instance, it does not make much sense to consider the neutral current as the carrier of static tonal attraction. For further examples illustrating the role of music for mathematics and physics – from Pythagoras to String Theory – we refer to Mazzola (2019).

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[3] Let us shortly discuss another point that is exciting for most physicists. In particle physics, the masses of the gauge Bosons play an essential role. They determine the extremely small region of the configuration space where the weak interaction really can take place. It needs a special mechanism to give the three gauge bosons their masses. This mechanism is connected with the name 'Higgs' – a Scottish physicist who first described the precise subtle mechanism by which such particles can get their masses. The application of ideas of quantum fields to describe musical attraction does not require that *all aspects* of particle fields have their pendant in the musical domain. As mental constructs, they do not have spatial extension and the concept of tonal masses does not make any reasonable sense. Hence, the symmetry-breaking Higgs mechanism does not make sense in the domain of cognitive music theory.

Supplement III: Metaphoric Models of Tonal Forces

Several authors explicitly or implicitly use the ideas of musical movements and musical forces as conceptual metaphors in the sense of Lakoff and Johnson (1980). This means that the source domain of naïve (folk) physics is assumed to constitute a conceptual network establishing main propositions about physical movements and their causes – the physical forces. Analogical reasoning is then used to transfer the physical concepts to the goal domain of tonal music. In this way, it is possible to describe the most plausible expectations generates by a listener during the processing of tonal music. This includes expectations based on static and dynamic forces.

Larson (1997-98, 2004; 2012) is the most prominent author who develops this idea in detail. In particular, he proposed three musical forces that generate melodic completions. He calls these forces 'gravity', 'inertia', and 'magnetism', respectively. These forces relate to conceptual metaphors (Lakoff and Johnson 1980) and structure our musical thinking per analogy with falling, inert and attracting physical bodies. Hence, physical forces are represented in our naïve (common sense) physics or folk physics.

Larson (2012) gives some examples concerning ordinary discourses about music. They demonstrate the metaphorical potential of the three forces ('gravity', 'inertia', and 'magnetism'). GRAVITY: The soprano's *high* notes rang *above.* The rising melodic line *climbed higher*. MAGNETISM: The music is *drawn* to this stable note. The *leading tone* is *pulled* to the tonic. INERTIA: The accompanimental figure, once set in motion… . This dance rhythm generates such momentum that… (citations at the end of Sect. 8).

Hereafter, we will present the basic ideas of Steve Larson as published in his last book (Larson 2012). We think that this book gives the best presently available overview on the field of musical forces. And it provides a fair discussion on related proposals such as Narmour's (1992) implication-realization model, the model of Bharucha (1996), Lerdahls (2001) algorithm, and related ideas of Margulis (2003) and others.

Larson (2012) investigates the empirical hypothesis that the average rating of each of the investigated patterns is a function of the sum of musical forces acting on that pattern. To do so, a linear regression analysis is performed testing the following hypothesis for the "net force" F for a probe tone x as reflected by the ratings:

$$F(x) = w_G \cdot G(x) + w_M \cdot M(x) + w_I \cdot I(x) \qquad (1)$$

Hereby, w_G, w_M, and w_I are the corresponding weight factors of the three constraint functions. The constraint functions themselves reflect the intuitive content of the phenomenological forces. For instance, the constraint $G(x)$ for gravity gets the value 1 (0) if the probe tone x is lower (higher) than the preceding tone. Hence, the constraint for gravity prefers falling tones to rising ones. The results of the linear regression analysis for the investigated data (Larson and van Handel, 2005) are $w_G = 0.4$, $w_M = 0.1$, $w_I = 1.2$. The correlation between model and data is $r = 0.95$. This high r-value means that the three forces, taken together, can account for about 90% of the variance of the frequency data. The two weight factors for gravity and magnetism are each significantly different from zero (at a 0.1 % level), but the weight for inertia is not. Interestingly, other studies using other data sets (Larson 2002) give a different result: gravity and inertia both make significant contributions but magnetism does not. In the 2005 study an additional analysis was performed that included in addition to gravity, magnetism, and inertia, an extra factor signaling *the ending on tonic* (= $\hat{1}$) was introduced. In this case the correlation is still higher: $r = 0.977$, and the extra factor got a weight of 0.46. Interestingly, the other factors now get weights different from the former analysis: $w_G = 0.16$ (instead of 0.4), $w_M = 0.26$ (instead of 0.1), and $w_I = 1.2$ (as before). Hence, magnetism and inertia both make significant contributions to linear regression but gravity does not. This demonstrated that the contribution of single factors to the "net force" can be evaluated only when the full context of all involved factors is given.

Finally, we want to stress that even a high correlation value of the fit as found in the data analysis does not answer the fundamental question about constraint grounding. As we have seen, the addition of some extra factors can radically change the influence of other factors and can even marginalize some factors. Hence, a multiple regression analysis with a high overall correlation coefficient cannot be taken as argument that the involved factors are all substantiated and "symbolically grounded" in the sense of Harnad (1990). As such, we cannot expect that these factors play a causal role in explaining tonal attraction.

We think Larson (2012) was aware of these problems. Several of his careful analyses try to justify the special role of musical forces. This contrasts with alternative analyses by earlier authors. Further, Larson (2012) has investigated different variants of various factors (constraints) and he found how delicately the cognitive system reacts even on minimal variations.

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