Exercises

1 Representing Uncertainty

1.1 You flip a (fair) coin exactly two times. What is the probability to get *head* at least once?

1.2 There is an urn containing three white and two black balls. You grab two at random. What is the probability that you grab two white ones?

1.3 There is a urn containing two white and two black balls. You grab three at random. What is the probability that you grab two white ones and one black one? (not necessarily on this order)

1.4 There are three urns labelled one, two, three. These urns contain, respectively, three white and three black balls, four white and two black balls, and one white and two black balls. An experiment consists of selecting an urn at random, then drawing a ball from it. Find the probability of drawing a black ball.

1.5 In representing the prisoner puzzle we made use of the following set of possible worlds: W = {(a,b), (a,c), (b,c), (c,b)}, where (x,y) represents a world where prisoner x is pardoned and the guard says that y will be executed. In this representation of the puzzle the constraints
(i) the guard always says the truth, and
(ii) if prisoner A happens to be executed, the guard doesn't say it
are already expressed in the set W

Your task is to give another representation of the puzzle: Start with a set $\Omega = \{a,b,c\}\times\{a,b,c\}$. Again (x,y) represents a world where prisoner x is pardoned and the guard says that y will be executed.
(i) Give the sets of possible worlds realizing the propositions *lives-a, lives-b, lives-c, says-a, says-b, says-c* $\subseteq \Omega$.
(ii) Give an explicit representation of the additional constraints in terms of the propositions *lives-x, says-x* for x,y$\in\{a,b,c\}$. Make use of logical operators, e.g. *lives-c* $\rightarrow\neg$*say-c*
(iii) Show the content C of these constraints reduces the set Ω to the set W = {(a,b), (a,c), (b,c), (c,b)}.
Remark: The pair (Ω, C) is an epistemic space representing the prisoner puzzle.

1.6 Prove Fact 2: Let $\Sigma = (W, W^0)$ be an epistemic space, and $U,V \subseteq W$ propositions. Further define $U\rightarrow V =_{\text{def}} \neg U\cup V$: Then it holds:

(i) $\Sigma \Vdash \text{Know}_x(U)$ & $\Sigma \Vdash \text{Know}_x(V) \Leftrightarrow \Sigma \Vdash \text{Know}_x(U\cap V)$

(ii) $\Sigma \Vdash \text{Possible}_x(U)$ or $\Sigma \Vdash \text{Possible}_x(V) \Leftrightarrow \Sigma \Vdash \text{Possible}_x(U\cup V)$

(iii) $\Sigma \Vdash \text{Know}_x(U\rightarrow V)$ & $\Sigma \Vdash \text{Know}_x(U) \Rightarrow \Sigma \Vdash \text{Know}_x(V)$

(iv) $U\subseteq V$ & $\Sigma \Vdash \text{Know}_x(U) \Rightarrow \Sigma \Vdash \text{Know}_x(V)$

(v) $W^0\subseteq V \Rightarrow \Sigma \Vdash \text{Know}_x(V)$

1.7 Show that 2^W (the set of all subsets of W) is an algebra!

1.8 Show that the system $\{\{1,2\},\{1\},\varnothing\}$ does not form an algebra!

1.9 Show that lower and upper probability are dual, i.e. $\mathcal{P}_*(U)+\mathcal{P}^*(\neg U) = 1$

1.10 Set of probabilities. Suppose that a bag contains 10 marbles; 5 are known to be red, and the remainder are known to be blue or green, although the exact proportion of blue and green is not known. We take one marble out of the bag. What's its colour? Describe the situation by a set of probability functions. Calculate the lower and upper probabilities for getting a blue marble. Use the same scenario in order to calculate the inner and outer measure.

1.11 *Inner and outer measure*. Suppose that a bag contains 10 marbles; 5 are known to be red, and the remainder are known to be blue or yellow or green, although the exact proportion of blue, yellow and green is not known. What is the likelihood that a marble taken out of the bag is
(i) yellow? (ii) red or yellow (iii) yellow or green.

Hint: $W = \{R, B, Y, G\}$

1.12 Show that the inner and the outer measure are dual

1.13 Show that the following inequalities hold for inner and outer measure:
(i) $\mu_*(U) \leq \mu^*(U)$
(ii) If $U \subseteq V$ then $\mu_*(U) \leq \mu_*(V)$ and $\mu^*(U) \leq \mu^*(V)$ [monotonicity]
(iii) $\mu_*(U \cup V) \geq \mu_*(U)+\mu_*(V)$ for disjoint U,V [superadditivity]
(iv) $\mu^*(U \cup V) \leq \mu^*(U)+\mu^*(V)$ for disjoint U,V [subadditivity]

1.14 *Possibility measure*. Let W be a finite set of possible worlds and Poss a possibility measure satisfying Poss1-Poss3. Show that $\mathrm{Poss}(U \cup V) = \max(\mathrm{Poss}(U), \mathrm{Poss}(V))$ holds even if U and V are not disjoint!

1.15 Assume that Poss is defined on all subsets of $W = \{1, \ldots, 10\}$ and take $\mathrm{Poss}(U) = \max_{n \in U}(n/10)$ and stipulate $\mathrm{Poss}(\varnothing)=0$. Show that Poss is a possibility measure.

1.16 Prove Fact 11: $\mathrm{Nec}(U \cap V) = \min(\mathrm{Nec}(U), \mathrm{Nec}(V))$.

1.17 Show that $\mathrm{Nec}(U) \leq \mathrm{Poss}(U)$!

1.18 *Ranking functions*. Prove Fact 12: Ranking functions can be viewed as possibility measures. Given a ranking function κ define the possibility measure Poss_κ by taking $\mathrm{Poss}_\kappa(U) = 1/(1+\kappa(U))$.

1.19 Define an entailment relation between propositions (subsets of) a given set of possible worlds W: $U \models V$ iff$_{def}$ $\kappa(V) \leq \kappa(U)$. Does this relation satisfy the three Tarskian principles for a monotonic inference relation?

REFLEXIVITY: $U \models U$

CUT: if $U \models U'$ and $U \cap U' \models V$, then $U \models V$

MONOTONICITY: if $U \models V$, then $U \cap U' \models V$

2 Updating Beliefs

2.1 Let Σ be an epistemic space and define for each $U \subseteq W$: $\Sigma \Vdash U$ iff $\Sigma|U = \Sigma$. Show that (i) $\Sigma \Vdash U \cap V$ iff $\Sigma \Vdash U$ *and* $\Sigma \Vdash V$ and (ii) if $\Sigma \Vdash U \rightarrow V$ and $\Sigma \Vdash U$, then $\Sigma \Vdash V$. What about union $\cup$?

2.2 Show that $\Sigma \Vdash \text{Know}_x(U)$ iff $\Sigma \Vdash U$ (omniscience)

2.3 Bayes' theorem. You are a witness of a night-time hit-and-run accident involving a taxi in Athens. All taxis in Athens are blue or green. You swear, under oath, that the taxi was blue. Extensive testing shows that under the dim lighting conditions, discrimination between blue and green is 75% reliable. Calculate the probability that the colour for the taxi indeed was blue. Assume that that 9 out of 10 Athenian taxis are green.

Hint for Solution: B = taxi was blue, LB = taxi looked blue.
$\mu(\text{LB}|\text{B})$, $\mu(\neg\text{LB},\neg\text{B}) = 0.75$. $\mu(\text{B}|\text{LB}) = ?$

2.4 Suppose you have a urn with 100 coins. One of the coins is double-headed, all the rest are fair. A coin is picked from the urn. For whatever reason, you can only test the coin by flipping it and examining the coin (i.e., you can't simply examine both sides of the coin). In the worst case, how many tosses do you need before having a posterior probability for either h (the coin is fair) or h' (the coin is double headed) that is greater than 0.99? (I.e., what's the minimum number of tosses until that happens).
Hint for the solution: Calculate the minimum number n such that $\mu(h' \mid e)>0.99$ for a sequence e of n *heads*.

2.5 *Set of probabilities*. 4 tosses of a biased coin (either 1/10 or 9/10 for head): $W = \{h, t\}^4$. $\mathcal{P} = \{\mu_{1/10}, \mu_{9/10}\}$, where $\mu_\alpha(hhhhh) = \mu_\alpha(H^1) \ldots \mu_\alpha(H^4) = \alpha^4$, etc. Calculate $\mathcal{P}_*(H^4| \ H^1\& \ H^2\& \ H^3)$ and $\mathcal{P}^*(H^4| \ H^1\& \ H^2\& \ H^3)$. Is the result convincing from an intuitive point of view?

2.6 Show that Poss($\cdot$ ||U) and Poss($\cdot$ |U) are possibility measures.

2.7 *Conditioning ranking functions*. Prove that $\kappa(U|V) = \kappa(V|U) + \kappa(U) - \kappa(V)$