Beliefs and Defaults

Material used

- Halpern: Reasoning about Uncertainty, Chapter 8

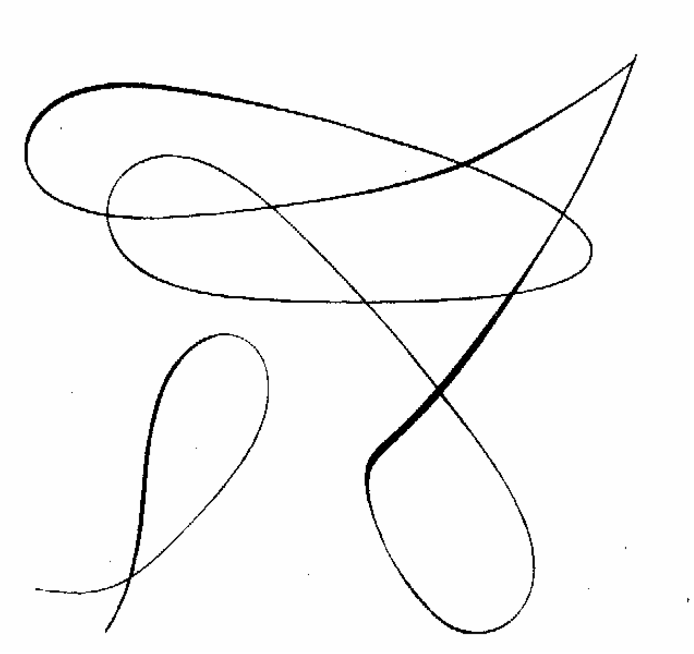
1 Motivating examples
2 Beliefs
3 Characterizing default reasoning
4 Semantics for defaults
5 Beyond system **P**

0 Introduction

- Default reasoning involves leaping to conclusions. It is a case of presumptive reasoning
- Counterfactual reasoning involves reaching conclusions with assumptions that may be counter to facts.
- Both cases of reasoning are defeasible and both involve uncertainty
- The earlier means to capture uncertainty (possibility, ranking functions and various measures for plausibility) can be used to define particular kinds of presumptive reasoning.

1 Some examples

Some examples should convince you that plausible or presumptive reasoning is not completely arbitrary but regulated by general principles.



- If birds typically fly, and if birds typically sing, then birds typically fly and sing
  $\{B>F, B>S\} \mid\!- B>(F \wedge S)$

- If red birds typically fly and if non-red birds typically fly, then birds typically fly (*reasoning by cases*)
  $\{(R \wedge B)>F, (\neg R \wedge B)>F\} \mid\!- B>F$

- * If birds typically fly, then handicapped birds typically fly (*monotonicity*)
  not valid: {B>F} |– (H∧B)>F
- * If birds typically fly, then penguins typically fly
  not valid: {B>F, P⇒B} |– P>F
- If birds typically fly and birds typically have wings, then birds that have wings typically fly (*cautious monotonicity*)
  {B>F, B>W} |– (B∧W)>F

Defeasible inferences

- If birds typically can fly and Fido is a bird, then Fido can fly (*defeasible modus ponens*)
  {B>F, B} |~ F
- *If birds typically can fly and Fido is a bird, but it cannot fly (it's a penguin), then Fido can fly
  not valid: {B>F, B, ¬F} |~ F; valid {B>F, B, ¬F} |~ ¬F
  [This shows that |~ is nonmonotonic]
- ?If students are typically adults and adults typically are car drivers, then students typically are car drivers (*transitivity*)
  ? {S>A, A>C} |~ S>C

2 Belief

A general model of beliefs uses *filters*.

Definition 1: Given a set of possible worlds W, a filter F is a nonempty set of subset of W that

1. is closed under supersets: $U \subseteq V \;\&\; U \in F \Rightarrow V \in F$
2. is closed under intersection: $U, V \in F \Rightarrow U \cap \mathrm{V} \in F$
3. does not contain the empty set

The general conception of a filter does not give any insight where beliefs are coming from. It's a descriptive modelling instrument only.

If $W^0 \subseteq W$ represents the agent's belief then we call (W, W^0) a *belief space*.

Definition 2: Let $\Sigma = (W, W^0)$ be a belief space.

- $\Sigma \Vdash \mathrm{Bel}_x(U)$ iff $W^0 \subseteq U$ (x beliefs U)

Fact 1: The events that are believed with regard to a fixed belief space Σ are filters, i.e.:

- $\Sigma \Vdash \mathrm{Bel}_x(U)$ & $U \subseteq V \Rightarrow \Sigma \Vdash \mathrm{Bel}_x(V)$
- $\Sigma \Vdash \mathrm{Bel}_x(U)$ & $\Sigma \Vdash \mathrm{Bel}_x(V) \Rightarrow \Sigma \Vdash \mathrm{Bel}_x(U \cap V)$
- not $\Sigma \Vdash \mathrm{Bel}_x(\emptyset)$

Let $(W, \mathcal{F}, \mu)$ be a probability space.

Fact 2: The events with probability 1 form a filter, i.e.

- $\mu(U)=1$ & $U \subseteq V \Rightarrow \mu(V)=1$
- $\mu(U)=1$ & $\mu(V)=1 \Rightarrow \mu(U \cap V)=1$
- $\mu(\emptyset)\neq 1$

In the following we investigate a more insightful model that makes use of plausibility spaces (generalizing probabilities)

- A plausibility measure is a generalization of all the approaches to uncertainty treated in the first part (probability, inner/outer measure, possibility, ranking functions)
- Formally, a plausibility space is a tuple S = (W, $\mathcal{F}$, Pl), where $\mathcal{F}$ is an algebra over W and Pl: $\mathcal{F} \rightarrow$ D where D is a set of plausibility values partially ordered by a relation $<_D$. The relation $<_D$ has a minimal element $\bot$ and a maximal element $\top$.
- Pl1. $Pl(\emptyset) = \bot$

  Pl2. $Pl(W) = \top$

  Pl3. $U \subseteq V \Rightarrow Pl(U) \leq Pl(V)$

With regard to a plausibility space S = (W, $\mathcal{F}$, Pl) it is possible to give the most general definition for beliefs.

Definition 3: Given a plausibility space S = (W, $\mathcal{F}$, Pl), say that an agent ***believes*** $U \in \mathcal{F}$ iff $\mathrm{Pl}(U) > \mathrm{Pl}(\neg U)$

Fact 3: This definition satisfies closure under supersets, i.e.:
$U \subseteq V$ & $\mathrm{Pl}(U) > \mathrm{Pl}(\neg U) \Rightarrow \mathrm{Pl}(V) > \mathrm{Pl}(\neg V)$

Proof: exercise

Unfortunately, this definition does not satisfy closure under conjunction in the general case:

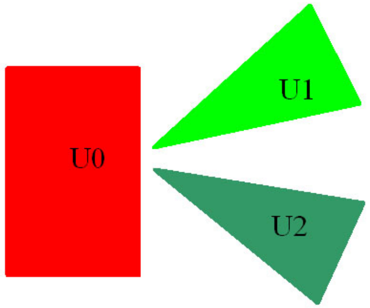
$$\mathrm{Pl}(U_1) > \mathrm{Pl}(\neg U_1) \;\&\; \mathrm{Pl}(U_2) > \mathrm{Pl}(\neg U_2) \;\Rightarrow\; \mathrm{Pl}(U_1 \cap U_2) > \mathrm{Pl}(\neg(U_1 \cap U_2))$$

However, we can stipulate this as an extra condition:

$\mathrm{Pl}(U_1) > \mathrm{Pl}(\neg U_1)$ & $\mathrm{Pl}(U_2) > \mathrm{Pl}(\neg U_2) \Rightarrow \mathrm{Pl}(U_1 \cap U_2) > \mathrm{Pl}(\neg(U_1 \cap U_2))$

In order to deal with conditioned plausibilities a somewhat stronger condition is stipulated:

Pl4. If U_0, U_1, and U_2 are pairwise disjoints sets, then $\mathrm{Pl}(U_0 \cup U_1) > \mathrm{Pl}(U_2)$ & $\mathrm{Pl}(U_0 \cup U_2) > \mathrm{Pl}(U_1)$ $\Rightarrow \mathrm{Pl}(U_0) > \mathrm{Pl}(U_1 \cup U_2)$



In words, if $U_0 \cup U_1$ is more plausible than U_2 and if $U_0 \cup U_2$ is more plausible than U_1, then U_0 by itself is already more plausible than $U_1 \cup U_2$.

Remark: Pl4 is necessary and sufficient to guarantee that (conditional) beliefs are closed under conjunction.

Ranking functions and the condition Pl4

Fact 4: The condition Pl4 is generally satisfied for possibility measures (and ranking functions).

In order to prove
$\mathrm{Pl}(U_0 \cup U_1) > \mathrm{Pl}(U_2)$ & $\mathrm{Pl}(U_0 \cup U_2) > \mathrm{Pl}(U_1) \Rightarrow \mathrm{Pl}(U_0) > \mathrm{Pl}(U_1 \cup U_2)$
assume that $\mathrm{Pl}(X \cup Y) = \max(\mathrm{Pl}(X), \mathrm{Pl}(Y))$

Case 1: $\mathrm{Pl}(U_0) \geq \mathrm{Pl}(U_1)$, then the premise reduces to $\mathrm{Pl}(U_0) > \mathrm{Pl}(U_2)$ & $\mathrm{Pl}(U_0) > \mathrm{Pl}(U_1)$ and the consequence part follows obviously.

Case 2.1: $\mathrm{Pl}(U_0) < \mathrm{Pl}(U_1)$, $\mathrm{Pl}(U_0) < \mathrm{Pl}(U_2)$, then the premise reduces to $\mathrm{Pl}(U_1) > \mathrm{Pl}(U_2)$ & $\mathrm{Pl}(U_2) > \mathrm{Pl}(U_1)$ and the consequence is trivially true.

Case 2.2: $\mathrm{Pl}(U_0) < \mathrm{Pl}(U_1)$, $\mathrm{Pl}(U_0) \geq \mathrm{Pl}(U_2)$, then the premise reduces to $\mathrm{Pl}(U_1) > \mathrm{Pl}(U_2)$ & $\mathrm{Pl}(U_0) > \mathrm{Pl}(U_1)$ and the consequence is true.

Consequence: *An agent believes U* according to definition 3 gives a *filter* if the plausibility function Pl is based on a possibility measure!

3 Characterizing default reasoning

Giving a set At of primitive (atomic) propositions, the language $\mathscr{L}^{\text{defaults}}(At)$ consists of all formulas of the form $\phi > \psi$ where ϕ and ψ are propositional formulas over At.

The formula $\phi > \psi$ can be read in various ways, depending on the application:

- If ϕ is the case then typically ψ is the case
- If ϕ the normally ψ
- If ϕ then by default ψ
- If ϕ then ψ is very likely
- If ϕ were the case then ψ would be true.

Core properties

Though there is some disagreement in the literature as to what properties > should have, there seems to be a consensus on the following set of six core properties, which make up the axiom system **P**:

- LLE (left logical equivalence): If $\phi \leftrightarrow \phi'$ is a propositional tautology, then from $\phi > \psi$ infer $\phi' > \psi$
- RW (right weakening): If $\psi \rightarrow \psi'$ is a propositional tautology, then from $\phi > \psi$ infer $\phi > \psi'$
- REF (reflexivity): $\phi > \phi$
- AND: From $\phi > \psi_1$ and $\phi > \psi_2$ infer $\phi > \psi_1 \wedge \psi_2$
- OR: From $\phi_1 > \psi$ and $\phi_2 > \psi$ infer $\phi_1 \vee \phi_2 > \psi$
- CM (cautious monotonicity): From $\phi > \psi_1$ and $\phi > \psi_2$ infer $\phi \wedge \psi_2 > \psi_1$

Definition 4:

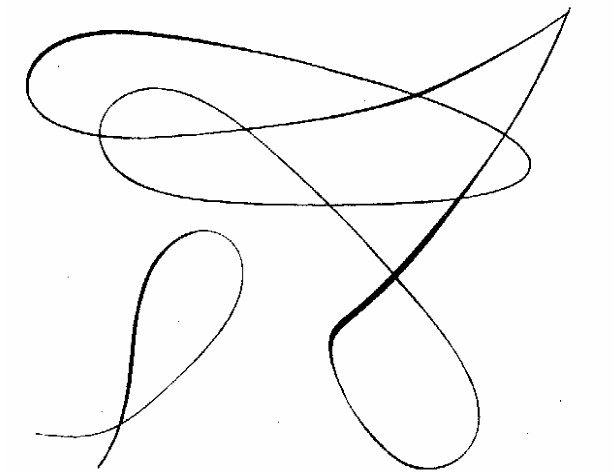
Let Σ be a finite set of formulas in $\mathscr{L}^{\text{defaults}}(At)$. Then write $\Sigma \mid\!-_{\mathbf{P}} \phi > \psi$ iff $\phi > \psi$ can be deduced from Σ using the rules and axioms of **P**

Example: Prove that {B>E, B>W, B>F} $\mid\!-_{\mathbf{P}}$ B$\wedge$(W$\vee$E) > F

1. Take {B>E, B>W, B>F} as premises
2. from 1 infer (B$\wedge$W) > F (CM)
3. from 1 infer (B$\wedge$E) > F (CM)
4. from 2 & 3 infer (B$\wedge$E)$\vee$(B$\wedge$W) > F (OR)
5. (B$\wedge$E)$\vee$(B$\wedge$W) $\leftrightarrow$ B$\wedge$(W$\vee$E) is a propositional tautology
6. from 4 & 5 infer B$\wedge$(W$\vee$E) > F (LLE)

4 Semantics for defaults

There have been many attempts to give semantics to formulas in $\mathscr{L}^{\text{defaults}}(At)$. The surprising thing is how many of them have ended up being characterized by the basic axiom system **P**. A semantics based on plausibility measures helps to explain why **P** characterizes so many different approaches. The property Pl4 is essential in this connection



Let $M = (W, \mu, \pi)$ be a simple probability structure, i.e. W is a set of possible worlds, μ a probability function on the subsets of W, and π is interpretation function for our language $\mathscr{L}^{\text{defaults}}(At)$. $\pi(p_i)$ assigns subsets of W to the atoms $p_i \in At$.

Definition 5: Interpretation of $\mathscr{L}^{\text{defaults}}(At)$.

$[\![p_i]\!] = \pi(p_i)$ for $p_i \in At$.

$[\![\neg\phi]\!] = \neg\, [\![\phi]\!]$

$[\![\phi\wedge\psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$

$[\![\phi\vee\psi]\!] = [\![\phi]\!] \cup [\![\psi]\!]$

Remark: all interpretations $[\![.]\!]$ are with regard to the structure M!

(CP) $M \models \phi > \psi$ iff $\mu(\, [\![\psi]\!] \mid [\![\phi]\!]\,) = 1$

- It is not difficult to show that this definition of defaults in (conditional) probability structures satisfies all the axioms and rules of axiom system **P**.
- In fact, P can be viewed as a sound and complete axiomatization of default reasoning for the language $\mathscr{L}^{\text{defaults}}(At)$. In order to make that precise…

Definition 6: $\Sigma \models \phi > \psi$ iff for all structures *M* for which each sentences of Σ is true, the default $\phi > \psi$ also is true
Fact 5: $\Sigma \vdash_{\mathbf{P}} \phi > \psi$ iff $\Sigma \models \phi > \psi$

- Is the intuitive interpretation of the last clause (CP) of Definition 5 really plausible?

Replace the condition (CP) in the definition 5 by the following definition making use of a fix, very small number $\varepsilon>0$.

(ε-CP) $M \models \phi > \psi$ iff $\mu([\![\psi]\!] \mid [\![\phi]\!]) > 1-\varepsilon$

- It can be shown this definition satisfies LLE, RW, REF but not AND, CM, and OR (see exercise)
- However, if we consider sequences of probability functions $(\mu_1, \mu_2, \ldots)$ then the corresponding definition conforms to **P**:

(∞-CP) $M \models \phi > \psi$ iff $\lim_{k\to\infty} \mu_k([\![\psi]\!] \mid [\![\phi]\!]) = 1$

- It is not so clear where the sequence of probabilities is coming from

(Poss) $M \models \phi > \psi$ iff Poss($[\![\phi]\!]$) = 0 or

Poss($[\![\phi\wedge\psi]\!]$) > Poss($[\![\phi\wedge\neg\psi]\!]$)

Remember the definition of possibility measures:

Poss1. Poss($\varnothing$)=0
Poss2. Poss(W)=1
Poss3. Poss($U\cup V$) = max(Poss(U), Poss(V))

Theorem: The definition (Poss) of the truth-conditions for $\phi > \psi$ satisfies all the axioms and rules of axiom system **P**. Moreover, **P** is a complete characterization of the corresponding semantics: $\Sigma \vdash_{\mathbf{P}} \phi > \psi$ iff $\Sigma \models \phi > \psi$.

(for the poof see Halpern, p. 299)

5 Beyond system P

- The system P has been viewed as characterizing the "conservative core" of default reasoning.
- For practical reasons (modeling of presumptive reasoning) it is useful to add a "nonmonotonic periphery" in order to deal with defeasible reasoning.
- One example is *defeasible modus ponens*, another is exceptional subclass inheritance:

  $\{penguin \Rightarrow bird, bird > winged\} \mid\!\sim penguin > winged$

  (Although penguins are an exceptional subclass of birds (property *fly*!) , it seems reasonable for them to still inherit the property of having wings from birds)

Instead of the standard definition of entailment (semantic consequence) repeated here a new definition 7 is used that makes use of preferred structures

Definition 6: $\Sigma \models \phi > \psi$ iff for **all** structures *M* for which each sentences of Σ is true, the default $\phi > \psi$ also is true

Definition 7: $\Sigma \models \phi > \psi$ iff for all **preferred** structures *M* for which each sentences of Σ is true, the default $\phi > \psi$ also is true

Example for preferred structures: looking for probability distributions that maximize the entropy (see Halpern p. 309)