Exercises

1 Representing Uncertainty

1.1 There is an urn containing three white and two black balls. You grab two at random. What is the probability that you grab two white ones?

1.2 There is a urn containing two white and two black balls. You grab three at random. What is the probability that you grab two white ones and one black one? (not necessarily on this order)

1.3 In representing the prisoner puzzle we made use of the following set of possible worlds: W = {(a,b), (a,c), (b,c), (c,b)}, where (x,y) represents a world where prisoner x is pardoned and the guard says that y will be executed. In this representation of the puzzle the constraints
(i) the guard always says the truth, and
(ii) if prisoner A happens to be executed, the guard doesn't say it
are already expressed in the set W

Your task is to give another representation of the puzzle: Start with a set $\Omega = \{a,b,c\}\times\{a,b,c\}$. Again (x,y) represents a world where prisoner x is pardoned and the guard says that y will be executed.
(i) Give the sets of possible worlds realizing the propositions *lives-a, lives-b, lives-c, says-a, says-b, says-c* $\subseteq \Omega$.
(ii) Give an explicit representation of the additional constraints in terms of the propositions *lives-x, says-x* for $x,y \in \{a,b,c\}$. Make use of logical operators, e.g. *lives-c* $\rightarrow\neg$*say-c*
(iii) Show the content C of these constraints reduces the set Ω to the set W = {(a,b), (a,c), (b,c), (c,b)}.
Remark: The pair (Ω, C) is an epistemic space representing the prisoner puzzle.

1.4 Prove Fact 2: Let $\Sigma = (W, W^0)$ be an epistemic space, and $U,V \subseteq W$ propositions. Further define $U \rightarrow V =_{\text{def}} \neg U \cup V$: Then it holds:

(i) $\Sigma \Vdash \text{Know}_x(U)$ & $\Sigma \Vdash \text{Know}_x(V) \Leftrightarrow \Sigma \Vdash \text{Know}_x(U \cap V)$

(ii) $\Sigma \Vdash \text{Possible}_x(U)$ or $\Sigma \Vdash \text{Possible}_x(V) \Leftrightarrow \Sigma \Vdash \text{Possible}_x(U \cup V)$

(iii) $\Sigma \Vdash \text{Know}_x(U \rightarrow V)$ & $\Sigma \Vdash \text{Know}_x(U) \Rightarrow \Sigma \Vdash \text{Know}_x(V)$

(iv) $U \subseteq V$ & $\Sigma \Vdash \text{Know}_x(U) \Rightarrow \Sigma \Vdash \text{Know}_x(V)$

(v) $W^0 \subseteq V \Rightarrow \Sigma \Vdash \text{Know}_x(V)$

1.5 Show that the system $\{\{1,2\},\{1\},\varnothing\}$ does not form an algebra!

1.6 Show that lower and upper probability are dual, i.e. $\mathcal{P}_*(U)+\mathcal{P}^*(\neg U) = 1$

1.7 Set of probabilities. Suppose that a bag contains 10 marbles; 5 are known to be red, and the remainder are known to be blue or green, although the exact proportion of blue and green is not known. We take one marble out of the bag. What's its colour? Describe the situation by a set of probability functions. Calculate the lower and upper probabilities for getting a blue marble. Use the same scenario in order to calculate the inner and outer measure.

1.8 *Inner and outer measure*. Suppose that a bag contains 10 marbles; 5 are known to be red, and the remainder are known to be blue or yellow or green, although the exact proportion of blue, yellow and green is not known. What is the likelihood that a marble taken out of the bag is
(i) yellow? (ii) red or yellow (iii) yellow or green.

Hint: $W = \{R, B, Y, G\}$

1.9 Show that the inner and the outer measure are dual

1.10 Show that the following inequalities hold for inner and outer measure:

(i) $\mu_*(U) \leq \mu^*(U)$

(ii) If $U \subseteq V$ then $\mu_*(U) \leq \mu_*(V)$ and $\mu^*(U) \leq \mu^*(V)$ [monotonicity]

(iii) $\mu_*(U \cup V) \geq \mu_*(U) + \mu_*(V)$ for disjoint U,V [superadditivity]

(iv) $\mu^*(U \cup V) \leq \mu^*(U) + \mu^*(V)$ for disjoint U,V [subadditivity]

1.11 *Possibility measure*. Let W be a finite set of possible worlds and Poss a possibility measure satisfying Poss1-Poss3. Show that $\text{Poss}(U \cup V) = \max(\text{Poss}(U), \text{Poss}(V))$ holds even if U and V are not disjoint!

1.12 Assume that Poss is defined on all subsets of $W = \{1, \ldots, 10\}$ and take $\text{Poss}(U) = \max_{n \in U} (n/10)$ and stipulate $\text{Poss}(\varnothing)=0$. Show that Poss is a possibility measure.

1.13 Prove Fact 11: $\text{Nec}(U \cap V) = \min(\text{Nec}(U), \text{Nec}(V))$.

1.14 Show that $\text{Nec}(U) \leq \text{Poss}(U)$!

1.15 *Ranking functions*. Prove Fact 12: Ranking functions can be viewed as possibility measures. Given a ranking function κ define the possibility measure Poss_κ by taking $\text{Poss}_\kappa(U) = 1/(1+\kappa(U))$.

2 Updating Beliefs

2.1 Let Σ be an epistemic space and define for each $U \subseteq W$: $\Sigma \Vdash U$ iff $\Sigma|U = \Sigma$. Show that (i) $\Sigma \Vdash U \cap V$ iff $\Sigma \Vdash U$ *and* $\Sigma \Vdash V$ and (ii) if $\Sigma \Vdash U \rightarrow V$ and $\Sigma \Vdash U$, then $\Sigma \Vdash V$. What about union $\cup$?

2.2 Show that $\Sigma \Vdash \mathrm{Know}_x(U)$ iff $\Sigma \Vdash U$ (omniscience)

2.3 Bayes' theorem. You are a witness of a night-time hit-and-run accident involving a taxi in Athens. All taxis in Athens are blue or green. You swear, under oath, that the taxi was blue. Extensive testing shows that under the dim lighting conditions, discrimination between blue and green is 75% reliable. Calculate the probability that the colour for the taxi indeed was blue. Assume that that 9 out of 10 Athenian taxis are green.

Hint for Solution: B = taxi was blue, LB = taxi looked blue.
$\mu(\mathrm{LB}|\mathrm{B})$, $\mu(\neg\mathrm{LB}|\neg\mathrm{B}) = 0.75$. $\mu(\mathrm{B}|\mathrm{LB}) = ?$

2.4 Suppose you have an urn with 100 coins. One of the coins is double-headed, all the rest are fair. A coin is picked from the urn. For whatever reason, you can only test the coin by flipping it and examining the coin (i.e., you can't simply examine both sides of the coin). In the worst case, how many tosses do you need before having a posterior probability for either *h* (the coin is fair) or *h'* (the coin is double headed) that is greater than 0.99? (I.e., what's the minimum number of tosses until that happens).

Hint for the solution: Calculate the minimum number n such that $\mu(h' \mid e) > 0.99$ for a sequence *e* of n *heads*.

2.5 *Set of probabilities*. 4 tosses of a biased coin (either 1/10 or 9/10 for head): $W = \{h, t\}^4$. $\mathcal{P} = \{\mu_{1/10}, \mu_{9/10}\}$, where $\mu_\alpha(hhhhh) = \mu_\alpha(H^1) \ldots \mu_\alpha(H^4) = \alpha^4$, etc. Calculate $\mathcal{P}_*(H^4 | H^1 \& H^2 \& H^3)$ and $\mathcal{P}^*(H^4 | H^1 \& H^2 \& H^3)$. Is the result convincing from an intuitive point of view?

2.6 Show that Poss($\cdot$ ||U) and Poss($\cdot$ |U) are possibility measures.

2.7 *Conditioning ranking functions*. Prove that $\kappa(V|U) = \kappa(U|V) + \kappa(U) - \kappa(V)$

3 Bayesian networks

3.1 Independence: Prove the equivalence of the following statements if $\mu(C) \neq 0$:

a. $\mu(B \cap C) \neq 0$ implies $\mu(A|B \cap C) = \mu(A|C)$

b. $\mu(A \cap C) \neq 0$ implies $\mu(B|A \cap C) = \mu(B|C)$

c. $\mu(A \cap B|C) = \mu(A|C)\,\mu(B|C)$

3.2 Prove that in a Bayesian network (with a defined Parent-function) the joint probability distribution can be calculated in the following way:

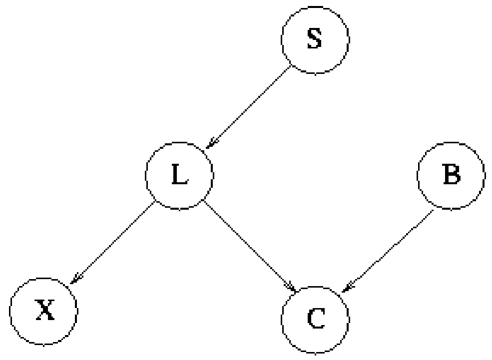
$$P(X_1, \ldots X_n) = \prod_{i=1}^{n} P(X_i \mid Parents(X_i))$$

3.3 One day Apple Jack discovers that his finest apple tree is losing its leaves. Now, he wants to know why this is happening. He knows that if the tree is dry (caused by a drought) there is no mystery - it is very common for trees to lose their leaves during a drought. On the other hand the losing of leaves can be an indication of a disease. How can this situation be modelled by a Bayesian network? Assume that the network consists of three nodes: **Sick**, **Dry**, and **Loses** which can all be in one of two states: Sick can be either "sick" or "not" - Dry can be either "dry" or "not" - and Loses can be either "yes" or "no". The node Sick tells us that the apple tree is sick by being in state "sick". Otherwise, it will be in state "not". The nodes Dry and Loses tell us in the same way if the tree is dry and if the tree is losing its leaves, respectively. Construct intuitively plausible (but still fictive) probability tables!

3.4 Prove that for any three 0-1-valued random variables P_1, P_2, P_3 on a probability space (W,μ) the following inequality holds (Bell's third variable inequality):

$$\mu(P_1=1, P_3=0) \leq \mu(P_1=1, P_2=0) + \mu(P_2=1, P_3=0).$$

3.5 Consider the following belief network for a medical diagnosis example, where B=Bronchitis, S=Smoker, C=Cough, X=Positive X-ray and L=Lung cancer. Suppose that the prior for a patient being a smoker is 0.25, and the prior for the patient having bronchitis (during winter in Melbourne!) is 0.05.



List the pairs of nodes that can be proven to be conditionally independent in the following situations (= given evidence):

a. There is no evidence for any of the nodes (absolute independence!).
b. The Lung cancer node is set to true (and there is no other evidence).
c. The Smoker node is set to true (and there is no other evidence).

d. The Cough node is set to true (and there is no other evidence).

Use (1) the definition of local semantics and (2) D-separation. Why do the outcomes differ in some cases?

3.6 Orville, the robot juggler, drops balls quite often when its battery is low. In previous trials, it has been determined that the probability that it will drop a ball when its battery is low is 0.9. On the other hand when its battery is not low, the probability that it drops a ball is only 0.01. The battery was recharged not so long ago, so there is only a 5% chance that the battery is low. A robot observer, with a somewhat unreliable vision system, reports on whether or not Orville has dropped the ball. This question involves constructing a belief network, containing only Boolean variables, to represent and draw inferences about whether the battery is low depending on how well Orville is juggling.

a. Draw a belief network to represent the problem. Label the network nodes and indicate clearly the direction of the arcs between the nodes.
b. Write down the probability tables showing where the information on how Orville's success is related to the battery level, and the robot observer's accuracy, are encoded in the network.
c. Suppose the robot observer reports that Orville has dropped the ball. What effect does this have on your belief that the battery is low. What type of reasoning is being done?

Hint: Use the nodes BL (**B**attery**L**ow), OD (**O**rville**D**rops), OSD (**O**bserver**S**ees **D**rop)

3.7 Prove that for any four 0-1-valued random variables A_1, A_2, B_1, B_2 on a probability space (W,μ) the following inequality holds (Bell's fourth variable inequality):

$$\mu(A_1=B_1) \leq \mu(A_1=B_2) + \mu(A_2=B_1) + \mu(A_2=B_2) .$$