Exercises

1 Dutch book

1.1 Assume that $U \cap V = \varnothing$, $\mu_X(U) = 0.4$, $\mu_X(V) = 0.5$, and $\mu_X(U \cup V) = 0.7$. Construct a Dutch book against X.

1.2 Assume that $U \cap V = \varnothing$, $\mu_X(U) = 0.2$, $\mu_X(V) = 0.4$, and $\mu_X(U \cup V) = 0.5$. Construct a Dutch book against X.

1.3 Assume that $\mu_X(U \cap V) = 0.2$, $\mu_X(U) = \mu_X(V) = 0.4$, and $\mu_X(U \cup V) = 0.7$. Construct a Dutch book against X.

2 Dempster-Shaver Theory

2.1 Calculate the plausibility function induced by m_1 for the safecracker example.

2.2 Show that for any $U, V \subseteq W$,
a. $\mathrm{Bel}(U \cup V) \geq \mathrm{Bel}(U) + \mathrm{Bel}(V) - \mathrm{Bel}(U \cap V)$
b. $\mathrm{Pl}(U \cap V) \leq \mathrm{Pl}(U) + \mathrm{PL}(V) - \mathrm{Pl}(U \cup V)$
Hint: Use the definitions of Bel and Pl in terms of mass functions

2.3 Let Γ be a function $H \Rightarrow 2^W - \{\varnothing\}$. Show that the following function is a mass function over W: $m_\Gamma(U) = \mu(\{h \in \mathbf{H}: \Gamma(h) = U\})$.
Show further that for the belief function Bel_Γ based on m_Γ, i.e. $\mathrm{Bel}_\Gamma(U) = \Sigma_{U' \subseteq U}\, m_\Gamma(U')$, the following identity is true: $\mathrm{Bel}_\Gamma(U) = \mu(\{h \in \mathbf{H}: \Gamma(h) \subseteq U\})$

2.4 Suppose that a bag contains 10 marbles; 5 are known to be red, and the remainder are known to be blue or yellow or green, although the exact proportion of blue, yellow and green is not known. First, define a mass function that describes this situation. Second, calculate the belief and plausibility function for the case that a marble taken out of the bag is (i) yellow? (ii) red or yellow (iii) yellow or green.
Hint: $W = \{R, B, Y, G\}$

2.5 Suppose that $W = \{1,2,3\}$. Define m as follows: $m(1) = .2$, $m(\{1,2\}) = 0.3$, $m(\{1,2,3\}) = 0.5$, $m(U) = 0$ if U is not one of $\{1\}$, $\{1,2\}$, $\{1,2,3\}$. Calculate Bel and Pl for all subsets of W!

2.6 Jaundice is a symptom of one of four diseases -- cirrhosis of the liver Ci, hepatitis H, gallstones G and cancer of the gall bladder Cg. Now Ci and H are liver problems, G and Cg are gall bladder problems. So you might have some evidence (sharp pain in the lower back, for instance) which was associated with liver problems. You might use this to define a probability assignment m_1 as follows: $m_1(\{Ci, H\}) = 0.7$, $m_1(\{Ci, H, G, Cg\}) = 0.3$.

[Comment: If m_1 had been an ordinary probability distribution then you would have expected $m_1(\{G, Cg\}) = 0.3$, which would have meant that a sharp pain in the lower back was evidence to degree 0.3 that the patient was **not** suffering from a liver problem. So DS probability assignments distribute the remaining belief over the universal hypothesis, whereas classical probability distributions distribute it over the complement of the current hypothesis.]

Now suppose that you have some other evidence (not a heavy drinker) that downgraded the hypothesis H. You can use this to define a probability distribution

m_2 as follows: $m_2(\{Ci, G, Cg\}) = 0.8$ and $m_2(\{Ci, H, G, Cg\}) = 0.2$.
a. Using Dempster's rule of combination, calculate the function $m_1 \oplus m_2$
b. Calculate the values of the belief/plausibility function for the arguments Ci, H, G, and Cg.

2.7 Let be $W = \{1,2,3,4,5,6\}$. The mass functions over W are as follows:
$m_1(\{1,2\}) = 1/6$, $m_1(\{3,4\}) = 1/3$, and $m_1(\{5,6\}) = ½$
$m_2(\{1\}) = 1/6$, $m_2(\{2\}) = 1/6$, $m_2(\{3\}) = 1/6$, $m_2\{4\}) = 1/6$, $m_2(\{5\}) = 1/6$, and $m_2(\{6\}) = 1/6$.

Calculate the mass function $m_1 \oplus m_2$, the belief function $Bel_1 \oplus Bel_2$, and the plausibility function $Pl_1 \oplus Pl_2$.

3 Fuzzy Logic

3.1 a. Prove that the support of $A \cap \neg A$ is empty iff $\forall x(\mu_A(x) = 0 \vee \mu_A(x) = 1)$.
b. Prove the corresponding for $A \cup \neg A$: the set of all x for which $A \cup \neg A$ is true is the whole domain V iff $\forall x(\mu_A(x) = 0 \vee \mu_A(x) = 1)$.

3.2 Consider an instance x of a "stripped apple" and demonstrate that $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$ doesn't make much sense if we consider this equation as a principle for determining degrees of prototypicality for conjoined concepts.

3.3 Prove that (a) xy is a t-norm, (b) min(x, y) is a t-norm; (c) max(0, x+y-1) is a t-norm. (d) what about $(xy)^2$?

3.4 Let A be a crisp set and B a crisp relation. Show that in this case relational *fuzzy* composition (version 1) agrees with *classical* composition $A \circ B = \{y: (\exists x) (x \in A \ \& \ [x,y] \in B)\}$.
Fuzzy composition 1: $\mu_{A \circ B}(y) = \sup_{x \in X}(\min(\mu_A(x), \mu_B(\langle x, y \rangle)))$

3.5 Let A be the crisp set {a} where a is a positive real number. Let R be the fuzzy relation $\mu_R(x, y) = e^{-(x-y)^2}$. Give an explicit expression for the relational composition of A with R.

3.6 Let be B and C crisp relations. Show that in this case relational *fuzzy* composition (version 2) agrees with *classical* composition $B \circ C = \{y: (\exists y) ([x,y] \in A \ \& \ [y,z] \in C)\}$.
Fuzzy composition 2: $\mu_{B \circ C}(\langle x, z \rangle) = \sup_{y \in Y}(\min(\mu_B(\langle x, y \rangle), \mu_C(\langle y, z \rangle)))$

3.7 Let $\nu(A)$ and $\nu(B) \in \{0, 1\}$. Show that the truth values of A&B, $A \vee B$, and $A \rightarrow B$ in fuzzy logic are those of classical two-valued logic.

3.8 Show that in fuzzy logic $\neg A \vee B$ and $A \rightarrow B$ are not equivalent.

3.9 Is the *modus ponens* valid in fuzzy logic? Prove it or give a counterexample!

3.10 Prove that $(A_1, \ldots, A_n) \models_F B$ is a valid fuzzy consequence if and only if $(A_1 \wedge \ldots \wedge A_n) \rightarrow B$ is a valid sentence in fuzzy logic

3.11 Verify that the following predicate logical principles are valid in (predicate) fuzzy logic:

a. $\forall 1$: $(\forall x\, A(x)) \rightarrow A(s)$ (*s* a term substitutable for *x* in A)

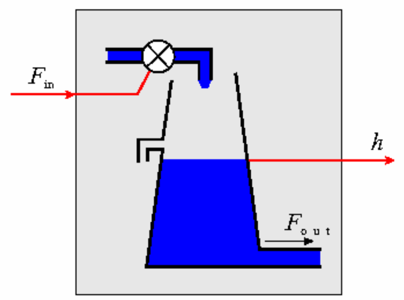
b. $\exists 1$: $A(s) \rightarrow (\exists x\, A(s))$ (*s* a term substitutable for *x* in A)

c. $\forall 2$: $\forall x\, (A \rightarrow B) \rightarrow (A \rightarrow \forall x\, B)$ (*x* not free in A)

d. $\exists 2$: $\forall x\, (A \rightarrow B) \rightarrow ((\exists x\, A) \rightarrow B)$ (*x* not free in B)

e. $\forall 3$: $\forall x\, (A \vee B) \rightarrow ((\forall x\, A) \vee B)$ (*x* not free in B)

3.12 Consider the following situation, which can be described by two rules:



R1: If FLOW is large then LEVEL is large
R2: If FLOW is small then LEVEL is small.

Assume $\mu_{large}(x) = x/10$ for $x \in [0, 10]$ and $\mu_{small}(y) = 1\text{-}y/10$ for $y \in [0, 10]$
What fuzzy set can be derived for the LEVEL if the FLOW is assumed to be in the interval [0,2] (a crisp set) ?

4 Quantum Probabilities

4.1. Let $\mathcal{H}$ be a Hilbert space and U, V sub (Hilbert) spaces of $\mathcal{H}$. Prove the following facts:

(a) $(U^\perp)^\perp = U$

(b) $(U \cap V)^\perp = U^\perp + V^\perp$

(c) $(U + V)^\perp = U^\perp \cap V^\perp$

4.2. In $\mathbb{R}_3$ assume a vector space U spanned by the unit vectors $u_1 = \frac{1}{\sqrt{2}}(1, 1, 0)$ and $u_2 = \frac{1}{\sqrt{2}}(1, -1, 0)$.

(a) Construct the corresponding projection operator, i.e. an operator $\hat{O}$ with the non-zero eigenvectors $\hat{O}(u_i) = u_i$ (i=1,2). Give the (3-dimensional) matrix representation of this operator (taking the basis vectors (1,0,0), (0,1,0), (0,0,1)).

(b) What form is the orthogonal operator $\hat{O}^\perp$?

(c) Show that the operators $\hat{O}$ and $\hat{O}^\perp$ commute, i.e. $[\hat{O}, \hat{O}^\perp] = 0$.

4.3. Word meanings and LSA.
Perform a SVD for the course example repeated here.

	Document 1	Document 2	Document 3
bank	0	0	1
bass	0.447	0.894	0
commercial	0	0.707	0.707
cream	1	0	0
guitar	1	0	0
fishermen	0	1	0
money	0	0.447	0.894

Present the matrix taking (a) the first three singular values vectors of the decomposition, (b) the first two singular values. (c) Calculate *bass* NOT *fisherman* in both cases and discuss the results.

4.4. Take the state described by the density matrix $\rho = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$ and calculate the expectations for the following projection operator (angel α polarizer):

$$\hat{P}_\alpha = \begin{pmatrix} \cos^2\alpha & \cos\alpha \cdot \sin\alpha \\ \cos\alpha \cdot \sin\alpha & \sin^2\alpha \end{pmatrix}.$$

Do the same for a state described by $\rho = ½\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

4.5. Show that for any density operator ρ that can be written in the form $\rho = |u\rangle\langle u|$, we get $\mathbf{Tr}(\rho\hat{O}) = \langle u|\hat{O}u\rangle$.

4.6. Calculate $\mu_\psi(\hat{P}_\alpha \hat{P}_\beta)$ for $|\psi\rangle = \frac{1}{\sqrt{2}}(|\leftrightarrow\rangle + |\updownarrow\rangle) = \frac{1}{\sqrt{2}}(1, 1)$. Take the angel α polarizer from exercise 4.4.

4.7. For which density matrixes (in a two dimensional Hilbert space) we get expectation values ½ for the two projectors $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$? Argue carefully why the solutions you are give are the only ones.